

Quantum Harmonic Black Holes

Roberto Casadio^{a,b*} and Alessio Orlandi^{a,b†}

^a*Dipartimento di Fisica e Astronomia, Università di Bologna
via Irnerio 46, 40126 Bologna, Italy*

^c*Istituto Nazionale di Fisica Nucleare, Sezione di Bologna
via Irnerio 46, 40126 Bologna, Italy*

April 5, 2013

Abstract

Inspired by the recent conjecture that black holes are condensates of gravitons, we investigate a simple model for the black hole degrees of freedom that is consistent both from the point of view of Quantum mechanics and of General Relativity. Since the two perspectives should “converge” into a unified picture for small, Planck size, objects, we expect our construction is a useful step for understanding the physics of microscopic, quantum black holes. In particular, we show that a harmonically trapped condensate gives rise to two horizons, whereas the extremal case (corresponding to a remnant with vanishing Hawking temperature) naturally falls out of its spectrum.

1 Introduction and motivation

One of the major mysteries in modern theoretical physics is to understand what are the internal degrees of freedom of black holes. This issue becomes particularly relevant in any attempt to develop a quantum theory which incorporates gravity along with the other forces of nature. Of course, without experimental inputs, our best starting point is the classical description of black holes provided by General Relativity [1], along with well established semiclassical results, such as the predicted Hawking radiation [2].

It was recently proposed by Dvali and Gomez that black holes are Bose-Einstein Condensates (BECs) of gravitons at a critical point, with Bogoliubov modes that become degenerate and nearly gapless representing the holographic quantum degrees of freedom responsible for the black hole entropy and the information storage [3]. In order to support this view, they consider a collection of objects (gravitons) interacting via Newtonian gravity,

$$V_N \sim -\frac{G_N \mu}{r} , \quad (1.1)$$

*roberto.casadio@bo.infn.it

†alessio.j.orlandi@gmail.com

and whose effective mass μ is related to their characteristic quantum mechanical size via the Compton/de Broglie wavelength,

$$\ell \simeq \frac{\hbar}{\mu} = \ell_p \frac{m_p}{\mu} . \quad (1.2)$$

These bosons can superpose and form a “ball” of radius ℓ , and total energy $M = N \mu$, where N is the total number of constituents. Within the Newtonian approximation, there is then a value of N for which the whole system becomes a black hole. In details, given the coupling constant

$$\alpha = \frac{\ell_p^2}{\ell^2} = \frac{\mu^2}{m_p^2} , \quad (1.3)$$

there exists an integer N such that no constituent can escape the gravitational well it contributed to create, and which can be approximately described by the potential

$$U(r) \simeq V_N(\ell) \simeq -N \alpha \frac{\hbar}{\ell} \Theta(\ell - r) , \quad (1.4)$$

where Θ is the Heaviside step function. This implies that components in the depleting region are “marginally bound”,

$$E_K + U \simeq 0 , \quad (1.5)$$

where the kinetic energy is given by $E_K \simeq \mu$. This energy balance yields the “maximal packing”

$$N \alpha = 1 . \quad (1.6)$$

Consequently, the effective boson mass and total mass of the black hole scale according to

$$\mu \simeq \frac{m_p}{\sqrt{N}} \quad \text{and} \quad M = N \mu \simeq \sqrt{N} m_p . \quad (1.7)$$

Note that one has here assumed the ball is of size ℓ (since bosons superpose) and, therefore, the constituents will interact at a maximum distance of order $r \sim \ell$, with fixed ℓ . The Hawking radiation and the negative specific heat spontaneously result from quantum depletion of the condensate for the states satisfying Eq. (1.5). This description is partly Quantum Mechanics and partly classical Newtonian physics, but no General Relativity is involved, in that geometry does not appear in the argument.

In this work, we will show how this picture, which draws from the conjectured UV-self-completeness of gravity [4], can be both improved within Quantum Mechanics and reconciled with the usual geometric description of space-time in General Relativity. Some considerations about the possible existence of remnants will also follow. We shall use units with $c = 1$, $\hbar = \ell_p m_p$ and the Newton constant $G_N = \ell_p/m_p$.

2 Quantum mechanical model

To summarise, Ref. [3] assumes that a black hole is a BEC, trapped in a gravitational well described by the simple potential (1.4). We can improve on this description, by employing the Quantum Mechanical theory of the harmonic oscillator as a (better) mean field approximation for

the Newtonian gravitational interaction acting on each boson inside the BEC. The potential U in Eq. (1.4) is therefore replaced by ¹

$$\begin{aligned} V &= \frac{1}{2} \mu \omega^2 (r^2 - d^2) \Theta(d - r) \\ &\equiv V_0(r) \Theta(d - r) , \end{aligned} \quad (2.1)$$

and we further set $V(0) = U(0)$, so that

$$\frac{1}{2} \mu \omega^2 d^2 = N \alpha \frac{\hbar}{\ell} . \quad (2.2)$$

We also assume that the effective mass, length and frequency of a single graviton mode are related by $\mu = \hbar \omega = \hbar/\ell$, which immediately leads to

$$d = \sqrt{2 N \alpha} \ell = \sqrt{2 N} \ell_p . \quad (2.3)$$

If we neglect the finite size of the well, the Schrödinger equation in polar coordinates,

$$\frac{\hbar^2}{2 \mu r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi = (V_0 - E) \psi , \quad (2.4)$$

yields the well-known eigenfunctions

$$\psi_{nlm}(r, \theta, \phi) = \mathcal{N} r^l e^{-\frac{r^2}{2\ell^2}} {}_1F_1(-n, l + 3/2, r^2/\ell^2) Y_{lm}(\theta, \phi) , \quad (2.5)$$

where \mathcal{N} is a normalization constant, ${}_1F_1$ the Kummer confluent hypergeometric function of the first kind and $Y_{lm}(\theta, \phi)$ are the usual spherical harmonics. The corresponding energy eigenvalues are given by

$$\begin{aligned} E_{nl} &= \hbar \omega \left[2n + l + \frac{3}{2} - V(0) \right] \\ &= \hbar \omega \left[2n + l + \frac{1}{2} \left(3 - \frac{d^2}{\ell^2} \right) \right] , \end{aligned} \quad (2.6)$$

where n is the radial quantum number and l the angular momentum (not to be confused with ℓ). Following the idea in Ref. [3], we view the above spectrum as representing the effective Quantum Mechanical dynamics of depleting modes, which can be described by the first (non-rotating) excited state ²

$$\psi_{100}(r) = \sqrt{\frac{2}{3 \ell^7 \sqrt{\pi}}} e^{-\frac{r^2}{2\ell^2}} (2r^2 - 3\ell^2) . \quad (2.7)$$

The marginally binding condition (1.5), that is $E_{10} \simeq 0$, then leads to the scaling laws

$$\ell = \sqrt{\frac{2N}{7}} \ell_p \quad \text{and} \quad \mu = \sqrt{\frac{7}{2N}} m_p , \quad (2.8)$$

¹This is nothing but Newton oscillator, which would correspond to a homogenous BEC distribution in the Newtonian approximation.

²Note we have already integrated out the angular coordinates.

in perfect qualitative agreement with Eq. (1.7).

We can now estimate the effect of the finite width of the potential well (2.1) by simply applying first order perturbation theory and obtain

$$\begin{aligned}\Delta E_{10} &= - \int_d^\infty r^2 dr \psi_{100}^2(r) V_0(r) \\ &\simeq - \frac{0.1}{\sqrt{N}} m_p .\end{aligned}\tag{2.9}$$

This can now be compared, for example, with the ground state energy $E_{00} = -\sqrt{14/N} m_p \simeq -3.7 m_p/\sqrt{N}$. Since $|\Delta E_{10}| \ll |E_{00}|$, our approximation appears reasonable.

We however remark that the ground state energy in this model has no physical meaning. Indeed, the Schrödinger equation (2.4) must be viewed as describing the effective dynamics of black hole constituents, and the total energy of the “harmonic black hole” is still given by the sum of the individual boson effective masses,

$$M = N \mu \simeq \sqrt{\frac{7N}{2}} m_p ,\tag{2.10}$$

in agreement with the “maximal packing” of Eq. (1.7) and the expected mass spectrum of quantum black holes (see, for example, Refs. [5, 6]).

3 Regular geometry

It is now reasonable to assume that the actual density profile of the BEC gravitational source is related to the ground state wave function in Eq. (2.5) according to

$$\rho(r) \simeq M \psi_{000}^2 \simeq \frac{7^2 m_p e^{-\frac{7r^2}{2N\ell_p^2}}}{\sqrt{\pi} N \ell_p^3} .\tag{3.1}$$

Similar Gaussians profiles have been extensively studied in Refs. [7, 8], where it was proven that such densities satisfy the Einstein field equations with a “de Sitter vacuum” equation of state, $\rho = -p$, where p is the pressure. Curiously, BECs can display this particular equation of state [9]. This feature provides a connection between Quantum Mechanics and the geometrical description.

Let us indeed take the static and normalised, energy density profile of Ref. [7],³

$$\rho(r) = \frac{M e^{-\frac{r^2}{4\theta}}}{\sqrt{4\pi} \theta^{3/2}} ,\tag{3.2}$$

where $\sqrt{\theta}$ is viewed as a fundamental length related to space-time non-commutativity, and r is the radial coordinate such that the integral inside a sphere of area $4\pi r^2$,

$$M(r) = \int_0^r \rho(\bar{r}) \bar{r}^2 d\bar{r} = M \frac{\gamma(3/2, r^2/4\theta)}{\Gamma(3/2)} ,\tag{3.3}$$

³The squared length θ should not be confused with one of the angular coordinates of the previous expressions. Also, note ρ has already been integrated over the angles.

gives the total Arnowitt-Deser-Misner (ADM) mass M of the object for $r \rightarrow \infty$. In the above, $\Gamma(3/2)$ and $\gamma(3/2, r^2/4\theta)$ are the complete and upper incomplete Euler Gamma functions, respectively. This energy distribution then satisfies Einstein field equations together with the Schwarzschild-like metric

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega^2, \quad (3.4)$$

where

$$f(r) = 1 - \frac{2 G_N M(r)}{r}. \quad (3.5)$$

According to Ref. [7], one has a black hole only if the *mass-to-characteristic length* ratio is sufficiently large, namely for

$$M \gtrsim 1.9 \frac{\sqrt{\theta}}{G_N} = 1.9 m_p \frac{\sqrt{\theta}}{\ell_p} \equiv M_*. \quad (3.6)$$

If the above inequality is satisfied, the metric function $f = f(r)$ has two zeros and there are two distinct horizons. For $M = M_*$, $f = f(r)$ has only one zero which corresponds to an “extremal” black hole, with two coinciding horizons (and vanishing Hawking temperature). The latter represents the minimum mass black hole, and a candidate black hole remnant of the Hawking decay [10]. Further, the classical Schwarzschild case is precisely recovered in the limit $G_N M/\sqrt{\theta} \rightarrow \infty$, so that departures from the standard geometry become quickly negligible for very massive black holes.

Going back to the BEC model, whose total ADM mass is given in Eq. (2.10), and comparing the Gaussian profile (3.1) with Eq. (3.2), that is setting $\theta = N \ell_p^2/14$, one finds that the condition in Eq. (3.6) reads

$$1.8 \sqrt{N} \gtrsim 0.5 \sqrt{N}, \quad (3.7)$$

and is always satisfied (for $N \geq 1$). We can therefore conclude that harmonic black holes always have two horizons, and the degenerate case is not realised in their spectrum. Although this mismatch might appear as a shortcoming of our construction, it is actually consistent with the idea that the extremal case should have vanishing Hawking temperature and therefore no depleting modes. It also implies that the final evaporation phase, if it ends in the extremal case, must be realised by a transition that most likely drives the BEC out of the critical point. The precise nature of such a “quantum black hole” state remains, however, unclear (see, for example, Refs. [11]).

4 Conclusions and outlook

We have shown that the scenario of Ref. [3], in which black hole inner degrees of freedom (as well as the Hawking radiation) correspond to depleting states in a BEC, can be understood and recovered in the context of General Relativity by viewing a black hole as made of the superposition of N constituents, with a Gaussian density profile, whose characteristic length is given by the constituents’ effective Compton wavelength. From the point of view of Quantum Mechanics, such states straightforwardly arise from a binding harmonic oscillator potential. Moreover, requiring the existence of (at least) a horizon showed that the extremal case, corresponding to a remnant with vanishing Hawking temperature, is not realised in the harmonic spectrum (2.10). Such states will therefore have to be described by a different model.

At the threshold of black hole formation (see, for example, Ref. [12] and References therein), for a total ADM mass $M \simeq m_p$ (thus $N \simeq 1$), the above description should allow us to describe Quantum Mechanical processes involving black hole intermediate (or metastable) states. In order to estimate the typical life-times of such quantum black holes, a better approximation of the potential outside the characteristic size of the object will likely be needed ⁴. However, we can already anticipate that quantum black holes with spin should be relatively easy to accommodate in our description, by simply considering states in Eq. (2.5) with $l > 0$. This should allow us to consider more realistic quantum black hole formation from particle collisions, since particles most likely scatter with non-zero impact parameter.

Many questions are still left open. First of all, the discretisation of the mass has an important consequence in the classical limit. For example, let us look again at Eq. (2.10), and consider two non-rotating black holes with mass $M_1 = \sqrt{\frac{7}{2} N_1} m_p$ and $M_2 = \sqrt{\frac{7}{2} N_2} m_p$, where N_1 and N_2 are positive integers, which slowly merge in a head-on collision (with zero impact parameter). The resulting black hole should have a mass M which is also given by Eq. (2.10). However, there is in general no integer N_3 such that $\sqrt{N_3} = \sqrt{N_1} + \sqrt{N_2}$. It therefore appears that either the mass should not be conserved, $M \neq M_1 + M_2$, or the mass spectrum described by Eq. (2.10) is not complete. This problem, which is manifestly more significant for small black hole masses (or, equivalently, integers N), is shared by all those models in which the black hole mass does not scale exactly like an integer. If we wish to keep Eq. (2.10), or any equivalent mass spectrum, we might then argue that a suitable amount of energy (of order $M_1 + M_2 - M_3$) should be expelled during the merging, in order to accommodate the overall mass into an allowed part of the spectrum. In this case, one may also wonder if this emission can be thought of as some sort of Hawking radiation ⁵, or if it is completely different in nature.

Another issue regards the assumption in Eq. (3.1), i.e. the idea that the classical density profile corresponds to the square modulus of the (normalised) wavefunction. At the semiclassical level, this seems reasonable and intuitive, but necessarily removes the concept of “point-like test particle” from General Relativity, thus forcing us to reconsider the idea of geodesics only in terms of propagation of extended wave packets, which might show unexpected features or remove others from the classical theory. Also, elementary particles would not differ from extended massive objects and therefore should have an equation of state (see, for instance, the old shell model in Refs. [14]). Would this equation of state be an observable and enter the description of the particle on the same level as any other quantum number? Do different particles have different equations of state?

Last but not least, there is the question of describing the formation of a BEC during a stellar collapse. Condensation is usually achieved at extremely low temperature, when the thermal de Broglie wavelength becomes comparable to the inter-particle spacing. Whereas one has no doubt that particles inside a black hole are extremely packed, it is not clear how such a dramatic drop of temperature could occur. One might find a reason for this in some modification of the laws of thermodynamics inside the event horizon.

⁴For example, one might adapt the construction yielding the effective potential acting on collapsing nested shells obtained in Refs. [13].

⁵Note that for vanishing impact parameter, one does not expect any emission of classical gravitational waves.

Acknowledgements

This work is supported in part by the European Cooperation in Science and Technology (COST) action MP0905 “Black Holes in a Violent Universe”.

References

- [1] S. Chandrasekhar, *The mathematical theory of black holes*, Oxford University Press (1992).
- [2] S.W. Hawking, *Nature* **248**, 30 (1974); *Commun. Math. Phys.* **43**, 199 (1975) [Erratum-ibid. **46**, 206 (1976)].
- [3] G. Dvali and C. Gomez, “Black Holes as Critical Point of Quantum Phase Transition”, arXiv:1207.4059 [hep-th]. “Black Holes 1/N Hair”, arXiv:1203.6575 [hep-th]. “Landau-Ginzburg Limit of Black Holes Quantum Portrait: Self Similarity and Critical Exponent”, arXiv:1203.3372 [hep-th]. “Black Hole Quantum N-Portrait”, arXiv:1112.3359 [hep-th].
- [4] G. Dvali and C. Gomez, “Self-Completeness of Einstein Gravity,” arXiv:1005.3497 [hep-th].
- [5] J.D. Bekenstein, *Lett. Nuovo Cim.* **11** (1974) 467.
- [6] G. Dvali, C. Gomez and S. Mukhanov, “Black Hole Masses are Quantized,” arXiv:1106.5894 [hep-ph].
- [7] P. Nicolini, A. Smailagic and E. Spallucci, *Phys. Lett. B* **632** (2006) 547.
- [8] P. Nicolini, *Int. J. Mod. Phys. A* **24** (2009) 1229; P. Nicolini, A. Orlandi and E. Spallucci, “The final stage of gravitationally collapsed thick matter layers,” arXiv:1110.5332 [gr-qc].
- [9] L.P. Pitaevskii and S. Stringari, *Bose-Einstein condensation*, Oxford University Press (2003).
- [10] R. Casadio and P. Nicolini, *JHEP* **0811**, 072 (2008).
- [11] X. Calmet, W. Gong and S.D.H. Hsu, *Phys. Lett. B* **668**, 20 (2008); X. Calmet, D. Fragkakis and N. Gausmann, *Eur. Phys. J. C* **71**, 1781 (2011).
- [12] R. Casadio, O. Micu and A. Orlandi, *Eur. Phys. J. C* **72**, 2146 (2012).
- [13] G.L. Alberghi, R. Casadio, G. P. Vacca and G. Venturi, *Class. Quant. Grav.* **16**, 131 (1999); G.L. Alberghi, R. Casadio, G. P. Vacca and G. Venturi, *Phys. Rev. D* **64**, 104012 (2001).
- [14] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev. Lett.* **4** (1960) 375; R. Casadio, R. Garattini and F. Scardigli, *Phys. Lett. B* **679** (2009) 156.